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Quasi-measures and image transformations on completely regular spaces

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Abstract

A map $\rho: C(X) \rightarrow C(Y)$ is called quasi-linear if it is linear on each singly generated subalgebra of $C(X)$. We prove the representation theorem for positive quasi-linear functionals in terms of integration with respect to quasi-measures. We then consider positive quasi-linear maps in more generality and show that they can be described in terms of transformations of sets in the underlying spaces. We call these image transformations. It is shown how smoothness properties of the quasi-linear map are reflected in the corresponding image transformation and the simplifications that occur when the underlying spaces are compact.

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1. Introduction

Let X be a completely regular Hausdorff space and $C(X)$ the collection of all bounded real-valued continuous functions on X . For each $f \in C(X)$, we define the singly generated subalgebra associated with f by

$$A(f) = \{\phi \circ f: \phi \in C(\text{spt } f)\}$$

where $\text{spt } f = \overline{f(X)} \subseteq \mathbb{R}$.

Definition 1. A function $\rho: C(X) \rightarrow \mathbb{R}$ is called a positive quasi-linear functional on X if

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- (a) Whenever $f \geq 0$, we have $\rho(f) \geq 0$;
- (b) For each $f \in C(X)$, ρ is linear on $A(f)$.

In [4], Boardman proved a representation theorem for positive quasi-linear functionals that is similar in form to the classical Riesz Representation theorem for linear functionals. In this theorem, the role of measures is replaced by that of quasi-measures, which are defined as follows:

Definition 2. Let $\mathcal{Z}(X)$ denote the collection of zero-sets in X and $\mathcal{U}(X)$ the collection of co-zero sets. Let $\mathcal{A}(X) = \mathcal{Z}(X) \cup \mathcal{U}(X)$, the Baire quasi-algebra on X . A Baire quasi-measure on X is a function $\mu: \mathcal{A}(X) \rightarrow [0, \infty)$ such that

- (a) $\mu(\emptyset) = 0$;
- (b) If A and B are disjoint elements of $\mathcal{A}(X)$ with $A \cup B \in \mathcal{A}(X)$, then $\mu(A \cup B) = \mu(A) + \mu(B)$;
- (c) If $U \in \mathcal{U}(X)$, then

$$\mu(U) = \sup\{\mu(Z): Z \in \mathcal{Z}, Z \subseteq U\}.$$

In particular, from (b), we have that $\mu(A) + \mu(X \setminus A) = \mu(X)$ for $A \in \mathcal{A}(X)$. Also, using (b) and (c), we have

$$\text{If } A \subseteq B \text{ in } \mathcal{A}(X), \quad \text{then } \mu(A) \leq \mu(B).$$

For later convenience, we also define $\mathcal{A}^*(X)$ to be the collection of subsets of X which are either open or closed and call $\mathcal{A}^*(X)$ the Borel quasi-algebra on X . We use the notation $\mathcal{O}(X)$ for the open sets in X . By analogy, we may define Borel quasi-measures where closed regularity is assumed instead of zero-set regularity.

It is natural to ask whether every quasi-measure can be extended to be a measure. In [1], Aarnes gives an example of a quasi-measure on $X = [0, 1]^2$ that cannot be extended in this way. In [9], it is shown that a quasi-measure on X can be extended to a finitely additive Baire measure if and only if μ is sub-additive on $\mathcal{U}(X)$. In this regard, notice that quasi-measures are only assumed to be additive on *disjoint* subsets whose union is in $\mathcal{A}(X)$. Since $\mathcal{A}(X)$ is not an algebra of sets, subadditivity is *not* automatic.

In the next section, we reprove Boardman's representation theorem in a more direct way. Then we consider those maps between $C(X)$ spaces that are multiplicative on each $A(f)$. This leads naturally to the idea of an image transformation.

2. The representation theorem

Let μ be a quasi-measure on X . For each $f \in C(X)$ we define, for $\alpha \in \mathbb{R}$, $\check{f}(\alpha) = \mu(f^{-1}(-\infty, \alpha))$ and $\hat{f}(\alpha) = \mu(f^{-1}(-\infty, \alpha])$. Clearly, both \check{f} and \hat{f} are monotone non-decreasing functions on \mathbb{R} . Furthermore, if $f(X) \subseteq [-M, M]$, then $\check{f}(\alpha) = \hat{f}(\alpha) = 0$ for $\alpha < -M$. Similarly, $\check{f}(\alpha) = \hat{f}(\alpha) = \mu(X)$ for $\alpha > M$.

We now define $\tilde{f}(\alpha) = \sup\{\check{f}(\beta): \beta < \alpha\}$. Notice that we get the same function here if we use \hat{f} in place of \check{f} in the definition. Also, \tilde{f} is now continuous from the left. Because of this, there exists a *measure*, μ_f on \mathbb{R} such that

- (a) $\mu_f(-\infty, \alpha) = \tilde{f}(\alpha) = \sup\{\mu(f^{-1}(-\infty, \beta]): \beta < \alpha\}$,
- (b) $\mu_f(-\infty, \alpha) = 0$ for $\alpha < -M$,
- (c) $\mu_f(-\infty, \alpha) = \mu(X)$ for $\alpha > M$,

where M is as above.

It is evident that

$$\mu_f(\alpha, \beta) = \sup\{\mu(f^{-1}[\gamma, \delta]): [\gamma, \delta] \subseteq (\alpha, \beta)\},$$

hence, for $U \subseteq \mathbb{R}$ open, we have

$$\mu_f(U) = \sup\{\mu(f^{-1}(K)): K \subseteq U \text{ is compact}\}.$$

We now define $\rho_\mu: C(X) \rightarrow \mathbb{R}$ by $\rho_\mu(f) = \int_{\mathbb{R}} i \, d\mu_f$ where i is the identity function on \mathbb{R} , i.e., $i(x) = x$. The task of this section is to show that ρ_μ is a positive quasi-linear functional on $C(X)$ and that every such functional is obtained from some quasi-measure. We will need the next lemma.

Lemma 3. *Let $f \in C(X)$ and $\phi \in C(\text{spt } f)$. Then $\mu_{\phi \circ f} = \phi^* \mu_f$, where $\phi^* \mu_f(U) = \mu_f(\phi^{-1}U)$.*

Proof.

$$\begin{aligned} \mu_{\phi \circ f}(U) &= \sup\{\mu(\phi \circ f)^{-1}(K): K \subseteq U \text{ is compact}\} \\ &= \sup\{\mu(f^{-1}(\phi^{-1}K)): K \subseteq U \text{ is compact}\} \\ &= \sup\{\mu(f^{-1}(L)): L \subseteq \phi^{-1}(U) \text{ is compact}\} \\ &= \mu_f(\phi^{-1}(U)) \\ &= \phi^* \mu_f(U), \end{aligned}$$

where we notice that $\phi^{-1}(K) \subseteq \text{spt } f$ is compact. \square

In particular, we have the following, very important equation.

$$\rho_\mu(\phi \circ f) = \int_{\mathbb{R}} \phi \, d\mu_f \tag{1}$$

for every $\phi \in C(\text{spt } f)$. In fact,

$$\rho_\mu(\phi \circ f) = \int_{\mathbb{R}} i \, d\mu_{\phi \circ f} = \int_{\mathbb{R}} i \, d\phi^* \mu_f = \int_{\mathbb{R}} \phi \circ i \, d\mu_f = \int_{\mathbb{R}} \phi \, d\mu_f.$$

It is now a simple matter to show the first part of the following theorem.

Theorem 4. Let μ be a positive quasi-measure on X . Then the function ρ_μ defined above is a positive quasi-linear functional on $C(X)$. Furthermore, every positive quasi-linear map can be obtained in this way from a unique quasi-measure.

Proof. First notice that if $f \geq 0$, then $\mu_f(-\infty, 0) \leq \mu f^{-1}(-\infty, 0) = 0$, so μ_f is concentrated on $[0, +\infty)$. Thus

$$\rho_\mu(f) = \int_{\mathbb{R}} i \, d\mu_f \geq 0.$$

Now, let $\phi \circ f, \psi \circ f \in A(f)$. Then

$$\begin{aligned} \rho_\mu(\alpha\phi \circ f + \psi \circ f) &= \rho_\mu((\alpha\phi + \psi) \circ f) \\ &= \int_{\mathbb{R}} \alpha\phi + \psi \, d\mu_f \\ &= \alpha \int_{\mathbb{R}} \phi \, d\mu_f + \int_{\mathbb{R}} \psi \, d\mu_f \\ &= \alpha\rho_\mu(\phi \circ f) + \rho_\mu(\psi \circ f), \end{aligned}$$

so ρ_μ is linear on $A(f)$. Thus, ρ_μ is a positive quasi-linear functional on $C(X)$.

For the converse, suppose that ρ is a quasi-linear functional on $C(X)$. We normalize ρ so that $\rho(1) = 1$. As in [1], it follows that $\rho(f) \leq \rho(g)$ when $f \leq g$.

For $U \in \mathcal{U}$, we define $\mu(U) = \sup\{\rho(f) : 0 \leq f \leq \chi_U\}$. Then we immediately see that for U_1, U_2 disjoint in \mathcal{U} , $\mu(U_1 \cup U_2) = \mu(U_1) + \mu(U_2)$. Also, $\mu(X) = 1$. Thus, it is consistent to define, for $Z \in \mathcal{Z}$, $\mu(Z) = 1 - \mu(X \setminus Z)$. We show that this defines a quasi-measure with $\rho = \rho_\mu$.

By linearity of ρ on $A(f)$, we have that $\rho(1 - f) = 1 - \rho(f)$, so that $\mu(Z) = \inf\{\rho(f) : \chi_Z \leq f\}$ for $Z \in \mathcal{Z}$. Thus, for $Z_1, Z_2 \in \mathcal{Z}$ disjoint, we have $\mu(Z_1) + \mu(Z_2) = \mu(Z_1 \cup Z_2)$. It also follows that $\mu(U) = \sup\{\mu(Z) : Z \in \mathcal{Z}, Z \subseteq U\}$ for $U \in \mathcal{U}$. Thus μ is a Baire quasi-measure on X .

Now fix $f \in C(X)$. Since ρ is linear on $A(f)$, there is a positive measure ν_f on $C(\text{spt } f)$ so that, for $\phi \in C(\text{spt } f)$,

$$\rho(\phi \circ f) = \int_{\mathbb{R}} \phi \, d\nu_f.$$

For fixed $\alpha \in \mathbb{R}$, let ϕ_n be defined to be 0 on $(-\infty, \alpha + 1/n]$, have value 1 on $[\alpha + 2/n, \infty)$, and be linear on $[\alpha + 1/n, \alpha + 2/n]$. Also, let $U(\beta) = f^{-1}(\beta, \infty)$.

Then, if $g \in C(X)$ with $0 \leq g \leq \chi_{U(\beta)}$ and $\beta > \alpha$, there is an n with $\phi_n \circ f \geq g$. Thus,

$$\begin{aligned} \mu_f(\alpha, \infty) &= \sup_{\beta > \alpha} \mu(f^{-1}(\beta, \infty)) \\ &= \sup_{\beta > \alpha} \sup_{0 \leq g \leq \chi_{U(\beta)}} \rho(g) \\ &= \sup_n \rho(\phi_n \circ f) \end{aligned}$$

$$\begin{aligned}
&= \sup_n \int_{\mathbb{R}} \phi_n \, d\nu_f \\
&= \nu_f(\alpha, \infty).
\end{aligned}$$

Since μ_f and ν_f are both measures on \mathbb{R} , this shows that $\mu_f = \nu_f$, so that $\rho_\mu(f) = \int i \, d\mu_f = \int i \, d\nu_f = \rho(f)$. Hence $\rho = \rho_\mu$. The uniqueness of μ is clear. \square

It should be pointed out that the representation of ρ_μ given here is the same as that in [4]. In fact, since $\check{f} = \tilde{f}$ a.e., we have

$$\rho_\mu(f) = \int i \, d\mu_f = M - \int_{-M}^M \check{f}(\alpha) \, d\alpha,$$

where $f(X) \subseteq [-M, M]$, which is equivalent to the description given by Boardman. From now on, we will write $\int_X f \, d\mu$ for $\rho_\mu(f)$.

Now, let X^* denote the collection of all $\{0, 1\}$ -valued quasi-measures on X . By [9], these quasi-measures correspond precisely to those quasi-linear maps from $C(X)$ into \mathbb{R} which are multiplicative on each subalgebra $A(f)$. We give X^* the weakest topology making all the maps $\mu \rightarrow \int f \, d\mu$ continuous for $f \in C(X)$. By an Alaoglu type argument, this makes X^* into a compact Hausdorff space.

As an alternative description of this topology, we notice that for each $U \in \mathcal{U}$, the set $\widehat{U} = \{\mu \in X^*: \mu(U) = 1\} = \bigcup_{f \leq \chi_U} \{\mu \in X^*: \int f \, d\mu > 0\}$ is open in X^* . On the other hand, the topology on X^* formed by making the sets $\{\widehat{U}: U \in \mathcal{U}\}$ a subbasis is Hausdorff, see [8], so it induces the same topology on X^* as that given above.

3. Image transformations

We now turn our attention to the properties of quasi-linear maps between $C(X)$ spaces. For the sake of simplicity of the theory, we restrict ourselves to those maps that are actually homomorphisms on singly generated subalgebras.

Definition 5. A map $\rho: C(X) \rightarrow C(Y)$ is called a quasi-homomorphism if it is linear and multiplicative on each subalgebra $A(f)$ for $f \in C(X)$. We also assume that $\rho(1) = 1$.

Notice that quasi-homomorphisms are automatically positive in the sense that $f \geq 0$ implies that $\rho(f) = (\rho(\sqrt{f}))^2 \geq 0$. Thus, for each $y \in Y$, there is a positive, normalized quasi-measure μ_y on X so that

$$\rho(f)(y) = \int_X f \, d\mu_y.$$

Since ρ is a quasi-homomorphism, we see that each μ_y defined above is $\{0, 1\}$ -valued and hence in X^* . Also, since $\rho(f) \in C(Y)$ for all $f \in C(X)$, the map $y \rightarrow \mu_y$ is continuous

from Y into X^* . Conversely, if $y \rightarrow \mu_y$ is a continuous map from Y into X^* , we obtain a quasi-homomorphism from $C(X)$ to $C(Y)$ by defining $\rho(f)(y) = \int f d\mu_y$.

Now suppose that $U \in \mathcal{U}$. The set \widehat{U} is open in X^* , so the set $q(U) = \{y \in Y: \mu_y(U) = 1\}$ is open in Y . Similarly, for $Z \in \mathcal{Z}$, the set $q(Z) = \{y \in Y: \mu_y(Z) = 1\}$ is closed in Y . In this way, we obtain a map $q: \mathcal{A}(X) \rightarrow \mathcal{A}^*(Y)$ from the Baire quasi-algebra on X to the Baire quasi-algebra on Y satisfying the following properties:

- (i) We have $q(X) = Y$;
- (ii) If $U \in \mathcal{U}(X)$, then $q(U) \in \mathcal{O}(Y)$;
- (iii) If A_1, A_2 are disjoint in $\mathcal{A}(X)$ with $A_1 \cup A_2 \in \mathcal{A}(X)$, then $q(A_1), q(A_2)$ are disjoint and $q(A_1 \cup A_2) = q(A_1) \cup q(A_2)$;
- (iv) If $A_1 \subseteq A_2$ in $\mathcal{A}(X)$, then $q(A_1) \subseteq q(A_2)$;
- (v) If $U \in \mathcal{U}(X)$, then $q(U) = \bigcup \{q(Z): Z \subseteq U, Z \in \mathcal{Z}(X)\}$.

We call a map satisfying the above conditions an *image transformation* from X to Y . More precisely, we will call such a map an image transformation from $\mathcal{A}(X)$ to $\mathcal{A}^*(Y)$. The appropriate definitions for other types of image transformations, say from $\mathcal{A}(X)$ to $\mathcal{A}(Y)$, are clear. Unless otherwise stated, we will assume an image transformation is as above. It is not clear whether an image transformation actually must have image in the Baire quasi-algebra, although as we shall see, many do so. We summarize the previous discussion in the following:

Theorem 6. *If $\rho: C(X) \rightarrow C(Y)$ is a quasi-homomorphism, then there is an image transformation $q: \mathcal{A}(X) \rightarrow \mathcal{A}^*(Y)$ such that $q(A) = \{y: \mu_y(A) = 1\}$, where $\rho(f)(y) = \int f d\mu_y$. Conversely, if $q: \mathcal{A}(X) \rightarrow \mathcal{A}^*(Y)$ is an image transformation, then there is a quasi-homomorphism $\rho: C(X) \rightarrow C(Y)$ satisfying the same relations.*

Under these circumstances, we also have

$$q(U) = \bigcup_{0 \leq f \leq \chi_U} \rho(f)^{-1}(0, +\infty).$$

Proof. We have already shown the forward direction of the first paragraph. But now, if q is an image transformation, define

$$\mu_y(A) = \begin{cases} 1 & \text{if } y \in q(A), \\ 0 & \text{if } y \notin q(A). \end{cases}$$

It is simple to check that μ_y is a $\{0, 1\}$ -valued quasi-measure, so the corresponding quasi-linear functional $f \rightarrow \int f d\mu_y$ is a quasi-homomorphism. If we now let $w: Y \rightarrow X^*$ be defined by $w(y) = \mu_y$, we then see that $w^{-1}(\widehat{U}) = q(U)$ is open, so w is continuous. Thus, if we define $\rho(f)(y) = \int f d\mu_y$, we find that $\rho(f) \in C(Y)$ for $f \in C(X)$ and that ρ is a quasi-homomorphism from $C(X)$ to $C(Y)$.

Finally, we have $y \in q(U)$ if and only if $\mu_y(U) = 1$, which is equivalent to the existence of an $0 \leq f \leq \chi_U$ such that $\rho(f) = \int f d\mu_y > 0$ since each μ_y is $\{0, 1\}$ -valued. \square

Corollary 7. *Let $\rho: C(X) \rightarrow C(Y)$ be a quasi-homomorphism. Then,*

- (i) If $f \leq g$ in $C(X)$, then $\rho(f) \leq \rho(g)$ in $C(Y)$.
- (ii) For any $f, g \in C(X)$, $\|\rho(f) - \rho(g)\|_\infty \leq \|f - g\|_\infty$.

Proof. Both of these results follow from the corresponding statements about quasi-measures when applied to μ_y . See [1] for proofs of these results. \square

It is known (see [9,8]) that if the covering dimension $\dim(X) \leq 1$, then every quasi-measure can be extended to a measure. Thus, we see that for such X , every quasi-homomorphism $C(X) \rightarrow C(Y)$ must be linear. More generally, we have the following.

Proposition 8. *Let $\rho : C(X) \rightarrow C(Y)$ be a quasi-homomorphism and q the corresponding image transformation. The following are equivalent:*

- (i) ρ is linear;
- (ii) $q(U_1 \cup U_2) = q(U_1) \cup q(U_2)$ for all $U_1, U_2 \in \mathcal{U}(X)$,

In this case, ρ is multiplicative on all of $C(X)$.

Proof. Suppose that ρ is linear. Then for each $y \in Y$, the quasi-measure μ_y is a measure and hence $\mu_y(U_1 \cup U_2) \leq \mu_y(U_1) + \mu_y(U_2)$. Since $q(U) = \{y \in Y : \mu_y(U) = 1\}$ and since each μ_y is $\{0, 1\}$ -valued, this establishes (ii). The converse is obtained by simply reversing this argument. For multiplicativity, we just notice that $fg = [(f + g)^2 - (f - g)^2]/4$. \square

As an example, if $g : Y \rightarrow X$ is a continuous function, then $q(A) = g^{-1}(A)$ defines an image transformation from X to Y . The corresponding quasi-linear map is given by $\rho(f) = f \circ g$ and is, in fact, linear. It is remarkable that every quasi-homomorphism can be written as the composition of a linear map and a quasi-homomorphism into a *fixed* space.

Proposition 9. *The map $q : \mathcal{A}(X) \rightarrow \mathcal{A}^*(X^*)$ defined by $q(A) = \widehat{A}$ is an image transformation. If $\rho : C(X) \rightarrow C(X^*)$ is the corresponding quasi-homomorphism, then $\rho(f)(\mu) = \int f d\mu$ for $\mu \in X^*$ and the functional ρ is norm-preserving. Finally, if $\rho' : C(X) \rightarrow C(Y)$ is any quasi-homomorphism, then ρ' factors as $L \circ \rho$, where $L : C(X^*) \rightarrow C(Y)$ is linear.*

Proof. That q is an image transformation is straightforward from its definition and the corresponding properties of $\{0, 1\}$ -valued quasi-measures. Independently, we see that ρ is a quasi-homomorphism by definition of the topology of X^* and the fact that integration against a $\{0, 1\}$ -valued quasi-measure is multiplicative on each subspace $A(f)$. Since

$$\begin{aligned}
 q(U) &= \{\mu : \mu(U) = 1\} \\
 &= \left\{ \mu : \text{there is a } 0 \leq f \leq \chi_U, \int f d\mu > 0 \right\} \\
 &= \left\{ \mu : \rho(f)(\mu) > 0 \text{ for some } 0 \leq f \leq \chi_U \right\}
 \end{aligned}$$

we see that q is the image transformation associated with ρ . Since we have, for $\mu \in X^*$,

$$\left| \int f d\mu \right| \leq \|f\| \mu(X) = \|f\|,$$

ρ is norm-decreasing. Since point masses are in X^* , the norm is actually preserved.

Now, suppose ρ' is any quasi-linear map, we let $w : Y \rightarrow X^*$ be the function defined by $w(y) = \mu_y$ where $\rho'(f)(y) = \int f d\mu_y$. Define $L : C(X^*) \rightarrow C(Y)$ by $L(g) = g \circ w$. Then L is linear and $\rho'(f)(y) = \int f d\mu_y = \rho(f)(\mu_y) = \rho(f)(w(y)) = L \circ \rho(f)(y)$. \square

Now suppose that $\rho : C(X) \rightarrow C(Y)$ is a quasi-homomorphism and that $\eta : C(Y) \rightarrow \mathbb{R}$ is a quasi-linear functional. It is clear that $\eta \circ \rho : C(X) \rightarrow \mathbb{R}$ is a quasi-linear functional. Now suppose that $q : \mathcal{A}(X) \rightarrow \mathcal{A}^*(Y)$ represents ρ as above and that ν is the quasi-measure associated with η in the representation theorem. We would like to have a description of the quasi-measure associated with $\eta \circ \rho$ in terms of q and ν . The following result does this.

Theorem 10. *Let $\rho : C(X) \rightarrow C(Y)$ be a quasi-homomorphism and $\eta : C(Y) \rightarrow \mathbb{R}$ a quasi-linear functional. Then the composition $\eta \circ \rho$ is a quasi-linear functional. Let $q : \mathcal{A}(X) \rightarrow \mathcal{A}^*(Y)$ be the image transformation corresponding to ρ and ν the quasi-measure corresponding to η . Let μ be the quasi-measure associated with $\eta \circ \rho$. Then*

$$\mu(U) = \sup \left\{ \nu(Z) : Z \in \mathcal{Z}(Y) \text{ such that there exists } Z' \in \mathcal{Z}(X), Z' \subseteq U, Z \subseteq q(Z') \right\}$$

for $U \in \mathcal{U}(X)$. We will write $\mu = q^* \nu$.

We thus see that whenever q is an image transformation from X to Y and ν is a quasi-measure on Y , the above formula defines a quasi-measure on X .

Proof. First notice that if $\chi_Z \leq f \leq \chi_U$, then $\chi_{q(Z)} \leq \rho(f) \leq \chi_{q(U)}$ where $Z \in \mathcal{Z}(X)$ and $U \in \mathcal{U}(X)$. In fact, if $y \in Y$, then $\chi_{q(Z)}(y) = \mu_y(Z) \leq \int f d\mu_y = \rho(f)(y)$. Similarly for the other inequality.

Now suppose that $\alpha < \mu(U)$. By regularity, there are $Z', Z'' \in \mathcal{Z}(X)$ and $U' \in \mathcal{U}(X)$ such that $Z' \subseteq U' \subseteq Z'' \subseteq U$ and $\mu(Z') > \alpha$. Choose $\chi_{Z'} \leq f \leq \chi_{U'}$ and let $V = \rho(f)^{-1}(0, \infty)$. Then $V \in \mathcal{U}(Y)$ and $\nu(V) > \int \rho(f) d\nu = \int f d\mu > \mu(Z') > \alpha$. By regularity of ν , there is then $Z \subseteq V$ such that $\nu(Z) > \alpha$. Hence $\mu(U)$ is no larger than the supremum in the statement of the theorem.

On the other hand, if $Z \in \mathcal{Z}(Y)$ with $Z \subseteq q(Z')$ where $Z' \subseteq U, Z' \in \mathcal{Z}(X)$, find $f \in C(X)$ with $\chi_{Z'} \leq f \leq \chi_U$. Then $\chi_Z \leq \rho(f)$, so $\nu(Z) \leq \int \rho(f) d\nu = \int f d\mu \leq \mu(U)$. \square

Clearly, if ν is $\{0, 1\}$ -valued, then so is $q^* \nu$, so we obtain a map $q^* : Y^* \rightarrow X^*$. Since this map is induced by composition with ρ , it is continuous. In this way, novel quasi-measures on X may be obtained. See the example in [7].

Given the asymmetric way that image transformations are defined, it is natural to ask when a transformation $q : \mathcal{A}(X) \rightarrow \mathcal{A}^*(Y)$ can be extended to a map with domain $\mathcal{A}^*(X)$ and to ask for situations when the image of q is contained in $\mathcal{A}(Y)$. Partial answers to both of these are readily obtained.

Proposition 11. *Suppose that X is a normal space. Then any image transformation $q : \mathcal{A}(X) \rightarrow \mathcal{A}^*(Y)$ can be extended to an image transformation with domain $\mathcal{A}^*(X)$ which is closed regular. In fact, the extension on closed sets is given by*

$$q(F) = \bigcap_{F \subseteq Z \in \mathcal{Z}(X)} q(Z).$$

Proof. For $y \in Y$, let μ_y be the quasi-measure on X associated with the quasi-homomorphism induced by q . According to [9], each μ_y may be extended to a Borel quasi-measure, μ_y^* , in a unique way by defining, for $F \subseteq X$ closed,

$$\mu_y^*(F) = \inf\{\mu_y(Z) : F \subseteq Z, Z \in \mathcal{Z}(X)\}.$$

We define the extension q^* on closed sets by $q^*(F) = \{y : \mu_y^*(F) = 1\}$. As long as this set is closed in Y , the quasi-measure properties of μ_y^* will guarantee that q^* is an image transformation with the required properties. However, we have that

$$q^*(F) = \bigcap_{F \subseteq Z \in \mathcal{Z}(X)} \{y : \mu_y(Z) = 1\}$$

is the intersection of closed sets, and hence is closed. \square

The question of whether the range of an image transformation is contained in $\mathcal{A}(Y)$ seems to be related to the smoothness properties of the corresponding quasi-homomorphism. Thus, we make the following definitions.

Definition 12. A quasi-homomorphism $\rho : C(X) \rightarrow C(Y)$ is said to be strongly σ -smooth if whenever $\{f_n\}$ is a sequence of functions in $C(X)$ with $f_n \downarrow f$ pointwise, then $\rho(f_n) \downarrow \rho(f)$.

Definition 13. An image transformation $q : \mathcal{A}(X) \rightarrow \mathcal{A}^*(Y)$ is said to be strongly σ -smooth if whenever $\{Z_n\}$ is a decreasing sequence in $\mathcal{Z}(X)$ with $Z_n \downarrow Z$, then $q(Z_n) \downarrow q(Z)$.

We note that similar definitions can be made for the other smoothness properties mentioned in [4]. We shall forgo this exposition since nothing novel is obtained once the case of strong σ -smoothness is understood.

Proposition 14. *Let $\rho : C(X) \rightarrow C(Y)$ be a quasi-homomorphism and q the corresponding image transformation. Then ρ is strongly σ -smooth if and only if q is strongly σ -smooth. Furthermore, in this case, the range of q is contained in the Baire quasi-algebra.*

Proof. Suppose that ρ is strongly σ -smooth and let $\mu_y \in X^*$ be such that $\rho(f)(y) = \int f d\mu_y$ for $f \in C(X)$. Then μ_y is strongly σ -smooth, that is $\mu_y(Z_n) \rightarrow \mu_y(Z)$ whenever $Z_n \downarrow Z$. Since $q(Z) = \{y \in Y : \mu_y(Z) = 1\}$, we have that q is strongly σ -smooth. Clearly, each step of this deduction may be reversed.

Now suppose, further that $Z \in \mathcal{Z}(X)$. Choose $Z_n \in \mathcal{Z}(X)$ and $U_n \in \mathcal{U}(X)$ such that $Z \subseteq Z_n \subseteq U_n \subseteq Z_{n-1}$ for all n and $Z_n \downarrow Z$. Now pick $f_n \in C(X)$ such that $f_n = 1$ on Z_n and $f_n = 0$ off U_n . Then, since each μ_y is strongly σ -smooth,

$$\mu_y(Z) = \lim \int f_n d\mu_y = \lim \rho(f_n)(y)$$

for all $y \in Y$. In particular,

$$q(Z) = \bigcap_n \{y: \rho(f_n)(y) = 1\}$$

is a zero-set in Y . \square

We note that we actually also showed that ρ is strongly σ -smooth if and only if each of the corresponding quasi-measures μ_y is strongly σ -smooth. In particular, since every quasi-measure on a pseudocompact space is strongly σ -smooth [4], the corresponding statement is true for image transformations.

We also notice that if μ is a strongly σ -smooth quasi-measure on X and $f \in C(X)$, there is a particularly simple description of the measure μ_f on \mathbb{R} that defines $\int f d\mu$. In this case,

$$\mu_f(\alpha, \beta) = \sup\{\mu f^{-1}[\gamma, \delta]: [\gamma, \delta] \subseteq (\alpha, \beta)\} = \mu f^{-1}(\alpha, \beta).$$

Thus, for any subset of \mathbb{R} which is either open or closed, $\mu_f(A) = \mu f^{-1}(A)$. This leads us to the following formula connecting ρ and q :

Corollary 15. *Let ρ be a strongly σ -smooth quasi-homomorphism and q the corresponding image transformation. For $f \in C(X)$, we have $q(f^{-1}(A)) = \rho(f)^{-1}(A)$ for $A \subseteq \mathbb{R}$ either open or closed.*

Proof. The proof consists of nearly untangling the definitions and noting that $(\mu_y)_f$ is the pointmass at $\rho(f)(y) = \int f d\mu_y$. \square

This is especially useful in the case when A is a singleton since it allows the computation of level sets of $\rho(f)$ from those of f . Now we show a particularly nice formulation of the image of a quasi-measure under an image transform in the case when both are strongly σ -smooth.

Corollary 16. *Suppose that $q: \mathcal{A}(X) \rightarrow \mathcal{A}^*(Y)$ is a strongly σ -smooth image transformation and ν a strongly σ -smooth quasi-measure on Y . Then $q^*\nu$ is a strongly σ -smooth quasi-measure on X and $q^*\nu(A) = \nu(q(A))$ for $A \in \mathcal{A}(X)$.*

Proof. Let ρ be the quasi-homomorphism corresponding to q and η the quasi-linear map corresponding to ν . Then ρ and η are both strongly σ -smooth, so the composition $\eta \circ \rho$ is also. Thus $q^*\nu$ is strongly σ -smooth.

Since $q(Z) \in \mathcal{A}(Y)$ for $Z \in \mathcal{A}(X)$, Theorem 10 shows that

$$q^*\nu(U) = \sup\{\nu(q(Z)): Z \in \mathcal{Z}(X), Z \subseteq U\} \leq \nu(q(U))$$

for $U \in \mathcal{U}(X)$. Fix U and select $Z_n \subseteq U_n \subseteq Z_{n+1} \subseteq U$ with $U_n \uparrow U$. Since q is strongly σ -smooth, we also have that $q(Z_n) \subseteq q(U_n) \uparrow q(U)$. Thus,

$$v(q(U)) = \lim_n v(q(U_n)) = \lim_n v(q(Z_n)) \leq q^*v(U) \leq v(q(U)).$$

That $q^*(Z) = v(q(Z))$ for $Z \in \mathcal{Z}(X)$ follows by taking complements. \square

4. Image transformations in compact spaces

Now we restrict ourselves to the case where X and Y are compact spaces. This situation is also discussed in another paper [3] which treats self-similarity from the point of view of image transformations and contains more examples. Then all quasi-measures are strongly σ -smooth and can be regarded as acting on the *Borel* quasi-algebra. We also have inner compact regularity for both quasi-measures and image transformations. The next result shows that image transformations actually preserve countable unions.

Proposition 17. *Let q be an image transformation between compact spaces. If $\{A_i\}_1^\infty$ is a disjoint family from $\mathcal{A}^*(X)$ whose union is in $\mathcal{A}^*(X)$, then*

$$q\left(\bigcup_{i=1}^\infty A_i\right) = \bigcup_{i=1}^\infty q(A_i).$$

Proof. This is a simple consequence of the countable additivity of quasi-measures on compact spaces, see [7]. \square

In a similar way, we may show that for a directed family of open sets $\{U_\alpha\}$,

$$q\left(\bigcup U_\alpha\right) = \bigcup q(U_\alpha).$$

Proposition 18. *Let $\rho: C(X) \rightarrow C(Y)$ be a quasi-homomorphism where X and Y are compact, and let $q: \mathcal{A}^*(X) \rightarrow \mathcal{A}^*(Y)$ be the corresponding image transformation. The following are equivalent:*

- (i) *For all $x \in X$, $q(\{x\}) \neq \emptyset$,*
- (ii) *q is one-to-one,*
- (iii) *ρ is an isometry,*
- (iv) *ρ is one-to-one.*

Proof. Suppose (i) and $A \neq B$ in $\mathcal{A}^*(X)$. Then there is an $x \in A \setminus B$. Then $\emptyset \neq q(\{x\}) \subseteq q(A) \setminus q(B)$, so $q(A) \neq q(B)$. Thus (ii) follows. Also (ii) implies (i) easily.

We now prove that (ii) implies (iii). Let $f, g \in C(X)$ and choose $x \in X$ with $\|f - g\|_\infty = |f(x) - g(x)|$. If $y \in q(\{x\})$, then $\mu_y(\{x\}) = 1$, so $\mu_y = \delta_x$, the pointmass at x . But then $|\rho(f)(y) - \rho(g)(y)| = |\int f d\mu_y - \int g d\mu_y| = |f(x) - g(x)| = \|f - g\|_\infty$, thus $\|\rho(f) - \rho(g)\|_\infty \geq \|f - g\|_\infty$. Since the opposite inequality is true for all quasi-homomorphisms, we have (iii). Clearly, (iii) implies (iv).

Now assume (iv) and let $x \in X$. The collection \mathcal{D} of closed neighborhoods of x is directed downward and has intersection $\{x\}$. Thus the intersection of $q(C)$ where C ranges over \mathcal{D} is $q(\{x\})$. If $q(\{x\}) = \emptyset$, compactness then guarantees a closed neighborhood C of x with $q(C) = \emptyset$. Choose $f \in C(X)$ with $\{x\} \prec f \prec C^\circ$. For any $\alpha > 0$, $\rho(f)^{-1}(\alpha) = q(f^{-1}(\alpha)) \subseteq q(C) = \emptyset$. Since $f \geq 0$, this shows that $\rho(f) = 0 = \rho(0)$, a contradiction. \square

Now we turn to the construction of examples of image transformations, and hence quasi-homomorphisms. Our first example gives a translation invariant quasi-measure on any compact group with a $\{0, 1\}$ -valued quasi-measure that is not a measure.

Example 1. Let G be a compact group and let μ be any $\{0, 1\}$ -valued quasi-measure on G that is not a measure. For example, we may take a quasi-measure on the square $[0, 1]^2$ and map it into the torus. Define an image transformation from G to G by setting $q(A) = \{g \in G: \mu(g^{-1}A) = 1\}$. We show that q is an image transformation. First, notice that if A and B are disjoint in $\mathcal{A}^*(G)$, then $q(A)$ and $q(B)$ must be disjoint. Otherwise, there would be $g \in G$ with $\mu(g^{-1}A) = \mu(g^{-1}B) = 1$ since $g^{-1}A$ and $g^{-1}B$ are disjoint, this contradicts that μ takes on only 0 and 1 as values. Furthermore, if $A \cup B \in \mathcal{A}^*(G)$, then $\mu(g^{-1}A \cup g^{-1}B) = 1$ which happens if and only if either $\mu(g^{-1}A) = 1$ or $\mu(g^{-1}B) = 1$. Thus, $q(A \cup B) = q(A) \cup q(B)$. Also, we see that if $A \subseteq B$, then $q(A) \subseteq q(B)$ and that $q(G) = G$. We still need to show inner compact regularity of q and that $q(U)$ is open if U is. The first is an easy consequence of regularity of μ . As for the second, suppose $g \in q(U)$. Pick a compact subset K of U with $\mu(g^{-1}K) = 1$. There is an open, symmetric, neighborhood V of the identity of G such that $VK \subseteq U$. Then for any $h \in Vg$, we have that $g^{-1}K \subseteq h^{-1}U$, so $\mu(h^{-1}U) = 1$. Thus $Vg \subseteq q(U)$, so $q(U)$ is open.

Now let λ be a left Haar measure on G and set $\nu = q^*\lambda$, so $\nu(A) = \lambda(q(A))$. Since $q(xA) = xq(A)$, we see that ν is left translation invariant. On the other hand, since μ is not a measure, $\mu(\{g\}) = 0$ for all $g \in G$. Since μ is $\{0, 1\}$ -valued, this implies some neighborhood, V_g of the identity that satisfies $\mu(gV_g) = 0$. By taking a finite subcover of $\{gV_g\}$ and letting V be the intersection of the V_x obtained, we get a neighborhood V of the identity such that $\mu(gV) = 0$ for all $g \in G$. But then, $q(V) = \emptyset$, so $\nu(V) = 0$. Thus ν is not a Haar measure on G .

Example 2. Let X and Y be compact and let μ be a $\{0, 1\}$ -valued quasi-measure on Y . We may define $\rho: C(X \times Y) \rightarrow C(X)$ by $\rho(f)(x) = \int f(x, y) d\mu(y)$. It is shown in [6] that ρ is a quasi-homomorphism and that $q(A) = \{x: \mu(U_x) = 1\}$ defines the corresponding image transformation. Here, $U_x = \{y: (x, y) \in U\}$.

We next turn to a more general procedure for producing examples of image transformations. To do so, we need some terminology and results from [2]. We assume that X is a compact, connected, locally connected space.

Definition 19. A subset A of X is said to be *solid* if both A and $X \setminus A$ are connected. We let $\mathcal{A}_s^*(X)$ denote those elements of $\mathcal{A}^*(X)$ which are solid with similar meanings for $\mathcal{C}_s(X)$ and $\mathcal{O}_s(X)$. A solid set function is a map $\mu: \mathcal{A}_s^*(X) \rightarrow [0, 1]$ such that

(i) If C_1, C_2, \dots, C_n are disjoint in $\mathcal{C}_s(X)$, then

$$\sum_1^n \mu(C_i) \leq 1;$$

(ii) If $U \in \mathcal{O}_s(X)$, then $\mu(U) = \sup\{\mu(C) : C \subseteq U, C \in \mathcal{C}_s(X)\}$;

(iii) If $\{A_i\}$ is a finite partition of X into elements of $\mathcal{A}_s^*(X)$, then

$$\sum \mu(A_i) = \mu(X) = 1.$$

From [2], we have the following important theorem.

Theorem 20. Assume that X is compact, connected and locally connected, and let μ be a solid set function on X . Then μ has a unique extension to a Borel quasi-measure on X . Furthermore, every Borel quasi-measure on X is obtained in this way.

In particular, a quasi-measure is determined by its values on solid sets. Using this and an argument as in [8], we see that the topology of X^* is generated by \widehat{U} where U ranges over the solid open sets in X . In fact, the only point that is tricky at all is that the topology induced by the solid sets is Hausdorff. But this follows from some basic facts about solid sets. See [2].

This allows us to immediately present a construction theorem for image transformations.

Theorem 21. Let X be compact, connected and locally connected and Y be any completely regular space. Assume that $q : \mathcal{A}_s^*(X) \rightarrow \mathcal{A}^*(Y)$ satisfies

- (a) If $C_1, C_2 \in \mathcal{C}_s(X)$ are disjoint, then $q(C_1)$ and $q(C_2)$ are disjoint;
- (b) If $U \in \mathcal{O}_s(X)$, then $q(U) = \bigcup \{q(C) : C \in \mathcal{C}_s(X), C \subseteq U\}$ is open;
- (c) If $\{A_i\}$ is a finite partition of X into sets from $\mathcal{A}_s^*(X)$, then $\bigcup q(A_i) = Y$.

Then q extends uniquely to an image transformation from $\mathcal{A}^*(X)$ to $\mathcal{A}^*(Y)$.

Proof. For $y \in Y$, define μ_y on $\mathcal{A}_s^*(X)$ by $\mu_y(A) = 1$ if $y \in q(A)$ and $\mu_y(A) = 0$ otherwise. Then μ_y is a solid set function, so extends uniquely to a $\{0, 1\}$ -valued quasi-measure on X which we also call μ_y . If we let $w : Y \rightarrow X^*$ be defined by $w(y) = \mu_y$, then w is continuous since for $U \in \mathcal{O}_s(X)$, the set $w^{-1}(\widehat{U}) = \{y : \mu_y(U) = 1\} = q(U)$ is open in Y by assumption. Hence, we may extend q to all of $\mathcal{A}^*(X)$ by setting $q(U) = w^{-1}(\widehat{U})$ for all $U \in \mathcal{O}(X)$ and obtain an image transformation. \square

For quite a large number of spaces, the third condition above trivializes to the simple condition that $q(X \setminus A) = Y \setminus q(A)$. In particular, if X is simply connected, this is the case. In essence, the *only* partitions of such a space X into elements of $\mathcal{A}_s^*(X)$ are the ones consisting of a set and its complement. See [5] for more details.

It is now an easy matter to give examples of a variety of image transformations.

Example. Let $X = Y = [0, 1]^2$. Let ∂ denote the boundary of the square and define, for $A \in \mathcal{A}_S^*$,

$$q(A) = \begin{cases} X & \text{if } \partial \subseteq A, \\ A & \text{if } \emptyset \neq A \cap \partial \neq \partial, \\ \emptyset & \text{if } A \cap \partial = \emptyset. \end{cases}$$

We then see that q may be extended to an image transformation. Some care is needed when calculating $q(A)$ when A is not solid, however. For example, let C be a closed, connected subset of X which intersects ∂ . Then $X \setminus C$ is the disjoint union of open, solid sets and $q(X \setminus C)$ is the union of those components of $X \setminus C$ which intersect ∂ . Thus $q(C)$ is the union of C with those components of $X \setminus C$ which do not intersect ∂ . Notice that if ρ is the quasi-homomorphism corresponding to q , Corollary 15 shows us that if f is constant on C , then $\rho(f)$ is constant on $q(C)$.

Example. Let X and Y be as above, and fix two points $x_1, x_2 \in X$. Define

$$q(A) = \begin{cases} X & \text{if } \text{card } A \cap \{x_1, x_2\} = 2, \\ A & \text{if } \text{card } A \cap \{x_1, x_2\} = 1, \\ \emptyset & \text{if } A \cap \{x_1, x_2\} = \emptyset. \end{cases}$$

Again, it is then easily seen that q extends to an image transformation. In this case, for C closed and connected, $q(C)$ is sensitive to whether both x_1 and x_2 are in the same component of $X \setminus C$. If they are not, $q(C)$ is the union of C with those components of $X \setminus C$ that contain neither x_1 nor x_2 .

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